

COINCIDENCE WECKEN PROPERTY FOR NILMANIFOLDS

DACIBERG GONÇALVES AND PETER WONG

ABSTRACT. Let $f, g : X \rightarrow Y$ be maps from a compact infra-nilmanifold X to a compact nilmanifold Y with $\dim X \geq \dim Y$. In this note, we show that the Wecken property holds, i.e., if the Nielsen number $N(f, g)$ vanishes then f and g are deformable to be coincidence free. We also show that if X is a connected finite complex X and the Reidemeister coincidence number $R(f, g) = \infty$ then $f \sim f'$ so that $C(f', g) = \{x \in X \mid f'(x) = g(x)\}$ is empty.

1. INTRODUCTION

Let $f, g : X \rightarrow Y$ be maps between topological spaces. The study of the coincidence set $C(f, g) = \{x \in X \mid f(x) = g(x)\}$ has long been a classical problem. Indeed in the 1920s, S. Lefschetz extended his celebrated fixed point theorem to coincidences. More precisely, he considered maps between closed connected orientable manifolds of the same dimension. A homological trace $L(f, g)$ is defined so that $L(f, g) \neq 0$ implies that $C(f, g) \neq \emptyset$. The converse of this result does not hold in general. In the fixed point case, a more subtle invariant, namely the Nielsen number $N(f)$ has been shown by F. Wecken that $N(f) = 0$ guarantees that f is deformable to be fixed point free when $X = Y$ is a compact manifold of dimension $\dim X \geq 3$. In other words, the Nielsen number $N(f)$ is a complete invariant for deforming selfmaps to be fixed point free.

In the same setting as that of Lefschetz, H. Schirmer successfully in 1955 generalized the Nielsen number to coincidences. Moreover, when $\dim X = \dim Y \geq 3$, she showed that $N(f, g) = 0 \Rightarrow f \sim f', g \sim g'$ such that $C(f', g') = \emptyset$. A more difficult situation is when $\dim X \geq \dim Y$. In recent years, progress has been made in such positive codimensional coincidence problem (see e.g. [8]). In fact, certain aspect of the coincidence problem turns out to be equivalent to the strong Kervaire invariant one problem ([3]).

When $g = \bar{c}$ is the constant map at $c \in Y$, the coincidence problem is equivalent to the root problem (since $C(f, g) = f^{-1}(c)$). A very general Wecken type theorem has been established in [4], namely, $N(f; c) = 0 \Rightarrow f \sim f'$ such that $f'^{-1}(c) = \emptyset$. Here $N(f; c)$ is the *geometric* Nielsen number studied by R. Brooks [2]. In [5], we showed that if $f, g : N_1 \rightarrow N_2$ are maps between two compact

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nilmanifolds with $\dim N_1 \geq \dim N_2$, then $N(f, g) > 0 \Rightarrow N(f, g) = R(f, g)$ where $R(f, g)$ is the coincidence Reidemeister number (the number of equivalence classes under $u \sim g_{\sharp}(x)uf_{\sharp}(x)^{-1}$ for $x \in \pi_1(N_1), u \in \pi_1(N_2)$) defined at the fundamental group level. Furthermore, if $N(f, g) > 0$ then $o_n(f, g) \neq 0$ where $n = \dim N_2$ and $o_n(f, g)$ is the primary obstruction to deforming f and g to be coincidence free on the n -th skeleton of N_1 .

The purpose of this note is to show that $o_n(f, g)$ is the *only* obstruction, i.e., $o_n(f, g) = 0 \Rightarrow f$ and g are deformable to be coincidence free. Equivalently, for maps between compact nilmanifolds, the Nielsen number $N(f, g)$ is a complete invariant. In fact, we are able to show that if X is a path connected topological space with finitely generated fundamental group and Y is a compact nilmanifold then for any two maps $f, g : X \rightarrow Y$, $R(f, g) = \infty \Rightarrow f \sim f', g \sim g'$ with $C(f', g') = \emptyset$. Finally, using a result of Brooks [1], one of the two homotopies can be made constant.

2. NIELSEN COINCIDENCE THEORY

For the coincidence problem where the codimension may be positive, we will make use of the more general (*geometric*) notion of the Nielsen coincidence number due to Brooks [2]. Given two maps $f, g : M \rightarrow N$, two coincidences $x_1, x_2 \in C(f, g)$ are *equivalent* if there is a path $\alpha : [0, 1] \rightarrow M$ with $\alpha(0) = x_1, \alpha(1) = x_2$ such that $f \circ \alpha$ is homotopic to $g \circ \alpha$ ($f \circ \alpha \sim g \circ \alpha$) relative to the endpoints. The equivalence classes are called coincidence classes. If $\{f_t\}, \{g_t\}$ are homotopies of f, g respectively, then $x \in C(f, g)$ and $y \in C(f_1, g_1)$ are $\{f_t\}, \{g_t\}$ -*related* if there exists a path $C : [0, 1] \rightarrow M$ with $C(0) = x, C(1) = y$ such that $[\{f_t \circ C(t)\}] = [\{g_t \circ C(t)\}]$ where $[f \circ C]$ denotes the fixed-endpoint homotopy class of the image of C under f . It follows that if a coincidence in a class γ of f, g is $\{f_t\}, \{g_t\}$ -related to a coincidence in a coincidence class β of f_1, g_1 , then every coincidence in γ is $\{f_t\}, \{g_t\}$ -related to every coincidence in β . In this case, the class γ is said to be $\{f_t\}, \{g_t\}$ -related to the class β . A coincidence class γ of f, g is *essential* if for any homotopies $\{f_t\}$ and $\{g_t\}$ of f and g , there is a coincidence class of f_1, g_1 to which it is related. Finally, the Nielsen coincidence number of f and g , denoted by $N(f, g)$, is defined to be the number of essential coincidence classes. This Nielsen number reduces to the classical one when the domain and the target have the same dimension.

As a consequence of the main theorem of [4], we obtain the following

Theorem 2.1. *Let $f, g : X \rightarrow M$ be maps from a compact topological space X to a compact connected topological group M . If $N(f, g) = 0$ then f and g are deformable to be coincidence free.*

Proof. It is straightforward to show that the coincidence problem is equivalent to the *root* problem concerning the preimage $\varphi^{-1}(e)$ where $e \in M$ is the group unity element and $\varphi(x) = [f(x)]^{-1} \cdot g(x)$ where \cdot denotes the group multiplication in M . Then $N(f, g) = 0$ iff $N(\varphi; \bar{e}) = 0$. The latter implies that φ can be made root free by the main theorem of [4]. \square

Remark 2.1. Theorem 2.1 also holds if $M = S^7$, the 7-sphere.

Theorem 2.2. *Let $f, g : N_1 \rightarrow N_2$ be maps between two compact nilmanifolds with $\dim N_1 \geq \dim N_2$. Then $N(f, g) = \emptyset$ if and only if $g \sim g'$ with $C(f, g') = \emptyset$.*

Proof. By the definition of $N(f, g)$, $C(f, g') \neq \emptyset$ for any $g' \sim g$ if $N(f, g) > 0$. For the converse, suppose $N(f, g) = 0$. It follows from [5, Theorem 4.2] that $R(f, g) = \infty$. Since N_2 is a nilmanifold, there is a principal S^1 -bundle $S^1 \hookrightarrow N_2 \xrightarrow{p} \bar{N}_2$. Without loss of generality, we may assume that $C(p \circ f, p \circ g)$ is a submanifold of dimension $> \dim N_1 - \dim N_2$. Note that $C(f, g) \subset C(p \circ f, p \circ g)$. Denote by \hat{f}, \hat{g} the restrictions of f, g to $C(p \circ f, p \circ g)$. Since the circle S^1 is a group, we have $\hat{g}(x) = d(x) \cdot \hat{f}(x)$ for $d : C(p \circ f, p \circ g) \rightarrow S^1$ and $x \in C(p \circ f, p \circ g)$. Here we call d the *deviation* map. We have two possibilities: d is null homotopic or d is NOT null homotopic.

If d is null homotopic then it is not difficult to show that the pair can be deformed to be coincidence free. So let us suppose that d is not null homotopic. Next, we will show that under this assumption, $R(f, g)$ must be finite. If N_2 is 2-dimensional then it is a torus and the theorem holds. We induct on $\dim N_2$. If $R(p \circ f, p \circ g)$ is infinite then by induction we are done. Suppose $R(\bar{f}, \bar{g})$ is finite where $\bar{f} = p \circ f, \bar{g} = p \circ g$.

Next, we show that the number of Reidemeister classes of f, g which project to the Reidemeister class $[\bar{1}]$ (where $\bar{1} \in \pi_1(\bar{N}_2)$) is finite. Let $x \in \pi_1(N_2)$ such that $p_\#(x) \in [\bar{1}]$. Then there exists $u \in \pi_1(N_1)$ such that $p_\#(x) = \bar{g}_\#(u) \bar{1} \bar{f}_\#(u)^{-1}$ which implies that $p_\#(g_\#(u)^{-1} x f_\#(u)) = 1$, i.e., the element $g_\#(u)^{-1} x f_\#(u)$ is in the same Reidemeister class of x and it belongs to the fundamental group of the fibre. For $v \in \pi_1(S^1)$, the elements in the Reidemeister class of v are of the form $g_\#(z) v f_\#(z)^{-1} = v g_\#(z) f_\#(z)^{-1}$ since $\pi_1(S^1)$ is central in $\pi_1(N_2)$. For a suitable z this element is an integer k due to the fact that the deviation map d is not null homotopic. In fact k is a generator of the image of $d_\#$. Thus the number of distinct Reidemeister classes is at most finite.

The map $d : C(p \circ f, p \circ g) \rightarrow S^1$ can certainly be extended to N_1 . To see that, note that the map d admits a lifting \hat{d} into \mathbb{R} . Then by Uryshon's Lemma \hat{d} admits an extension to N_1 . Let us consider the composite of the extension of \hat{d} with the projection $\exp : \mathbb{R} \rightarrow S^1$. Call d_e the composite.

Define $g'(x) = (\epsilon \cdot d_e(x)) \cdot g(x)$ where $\epsilon \in S^1 - \{1\}$. The extension d_e restricted to $C(p \circ f, p \circ g)$ is homotopic to a constant map at some $z \neq 1$ in S^1 . This partial homotopy admits an extension to a homotopy $H : N_1 \times I \rightarrow S^1$. Consider the map $g' = z \cdot H(\cdot, 1) \cdot g(\epsilon \cdot d_e(x))$ and $\cdot g(x)$ means the action of S^1 on the total space of the fibration.

The map g' is homotopic to g since the deviation is null homotopic. Moreover, $C(f, g') = \emptyset$. since g and g' differ by a deviation and the only possibility to have a coincidence point is if $x \in C(p \circ f, p \circ g)$. But for $x \in C(p \circ f, p \circ g)$, $g' = \epsilon \cdot f$ so there are no coincidence points and the result follows. \square

From the proof of Theorem 2.2, we summarize in the following theorem that certain conditions are equivalent.

Theorem 2.3. *For any two maps $f, g : N_1 \rightarrow N_2$ between two compact nilmanifolds with $\dim N_1 \geq \dim N_2$, the following are equivalent.*

- (1) $N(f, g) = 0$;
- (2) $R(f, g) = \infty$;
- (3) f and g are deformable to be coincidence free.

Corollary 2.4. Let $f, g : M \rightarrow N$ be two maps between two compact nilmanifolds with $\dim M \geq n = \dim N$. Then f and g are deformable to be coincidence free iff the (primary) obstruction $o_n(f, g) = 0$.

Proof. The result follows from Theorem 2.2 and [5, Theorem 4.2]. □

We now further extend Theorem 2.2 where the domain is arbitrary.

Theorem 2.5. *Let X be a connected topological space with finitely generated $\pi_1(X)$ and N a compact nilmanifold. For any maps $f, g : X \rightarrow N$, if $R(f, g) = \infty$ then f and g are deformable to be coincidence free.*

Proof. It follows from the proof of [6, Theorem 3] that there is a homotopy commutative diagram

$$(2.1) \quad \begin{array}{ccc} X & \xrightarrow{f, g} & N \\ \searrow q & & \nearrow \bar{f}, \bar{g} \\ & \bar{N} & \end{array}$$

where \bar{N} is a compact nilmanifold, $R(f, g) = R(\bar{f}, \bar{g})$ and $q_{\#} : \pi_1(X) \rightarrow \pi_1(\bar{N})$ is surjective. Since $R(f, g) = \infty$, it follows that $R(\bar{f}, \bar{g}) = \infty$. Thus, by Theorem 2.3, $N(\bar{f}, \bar{g}) = 0$ and \bar{f}, \bar{g} are deformable to be coincidence free. Note that the coincidences of f, g project to coincidences of \bar{f}, \bar{g} under q . We conclude that f, g are also deformable to be coincidence free. □

Remark 2.2. The converse of the above theorem does not hold in that f and g are deformable to be coincidence free while $R(f, g)$ is finite. For example, let S_2 be the orientable surface of genus 2, S^n the n -sphere with $n > 2$ and T^4 the 4-dimensional torus. Consider the map $\theta : S_2 \rightarrow T^4$ such that $\theta_{\#} : \pi_1(S_2) \rightarrow \pi_1(T^4)$ is the abelianization (thus a surjective induced homomorphism) and $c : S_2 \rightarrow T^4$ the constant map. Let $p : S_2 \times S^n \rightarrow S_2$ denote the projection on the first coordinate, $f = \theta \circ p$ and $g = c \circ p$. Since f and g factor through S_2 , which has dimension 2, it follows that θ and c are deformable to be coincidence free and so are f and g . It is straightforward to show that $R(f, g) = 1$.

Corollary 2.6. Let $f, g : M \rightarrow N$ be maps where M is a compact infra-nilmanifold and N a compact nilmanifold. Then $N(f, g) = 0$ iff $R(f, g) = \infty$. Moreover, if $N(f, g) > 0$ then $N(f, g) = R(f, g)$.

Proof. Let $p : \tilde{M} \rightarrow M$ be a finite covering where \tilde{M} is a compact nilmanifold. Furthermore, we may assume without loss of generality that this covering is regular. Define $\tilde{f}, \tilde{g} : \tilde{M} \rightarrow N$ by $\tilde{f} = f \circ p, \tilde{g} = g \circ p$. Then $N(f, g) = 0 \Rightarrow N(\tilde{f}, \tilde{g}) = 0$. To see this, first note that $p(C(\tilde{f}, \tilde{g})) \subseteq C(f, g)$. Let $\tilde{x} \in C(\tilde{f}, \tilde{g})$. Now let γ be the coincidence class of f and g containing $p(\tilde{x})$. Since $N(f, g) = 0$ there exist $\{f_t\}, \{g_t\}$ such that $p(\tilde{x})$ is not related to any coincidences of f_1 and g_1 . It is easy to see that \tilde{x} is not related to any coincidences of $\tilde{f}_1 \circ p$ and $\tilde{g}_1 \circ p$. It follows that $R(\tilde{f}, \tilde{g}) = \infty$ since \tilde{M} is a compact nilmanifold. Since $\pi_1(\tilde{M})$ is a finite index (normal) subgroup of $\pi_1(M)$, we have the following commutative diagram.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(\tilde{M}) & \xrightarrow{p_\#} & \pi_1(M) & \longrightarrow & F \longrightarrow 1 \\
 & & \tilde{f}_\# \downarrow \tilde{g}_\# & & f_\# \downarrow g_\# & & \downarrow \\
 1 & \longrightarrow & \pi_1(N) & \xrightarrow{=} & \pi_1(N) & \longrightarrow & 1 \longrightarrow 1
 \end{array}$$

Here, F denotes the finite quotient $\pi_1(M)/\pi_1(\tilde{M})$. It follows from [7, Theorem 2.1] that $R(f, g) = \infty = R(\tilde{f}, \tilde{g})$. Then the equivalence $N(f, g) = 0 \Leftrightarrow R(f, g) = \infty$ follows from Theorem 2.2. Now, suppose $N(f, g) > 0$. Then $R(f, g) < \infty$. It follows from the commutative diagram above (also [7, Theorem 2.1]) that $R(\tilde{f}, \tilde{g}) < \infty$ and hence every coincidence class of \tilde{f} and \tilde{g} must be essential. If $x \in C(f, g)$ then for any $\tilde{x} \in p^{-1}(x)$, $\tilde{x} \in C(\tilde{f}, \tilde{g})$. If x were to belong to an inessential class of f, g , it is easy to see that \tilde{x} would also have to belong to an inessential coincidence class of \tilde{f}, \tilde{g} , a contradiction since there are no such classes. We thus conclude that $N(f, g) = R(f, g)$ when $N(f, g) > 0$. \square

3. JIANG TYPE PROPERTY

In classical fixed point theory, selfmaps on Jiang spaces possess the following property: if $L(f) = 0$ then $N(f) = 0$; if $L(f) \neq 0$ then $N(f) = R(f)$. For coincidences, if M and N are closed orientable manifolds of the same dimension and N is either a nilmanifold [6] or a coset space of compact connected Lie group [9] then either $L(f, g) = 0 \Rightarrow N(f, g) = 0$ or $L(f, g) \neq 0 \Rightarrow N(f, g) = R(f, g)$. If $\dim M > \dim N$ and if both M and N are compact nilmanifolds then we have $N(f, g) = 0 \Leftrightarrow R(f, g) = \infty$ and $N(f, g) = R(f, g)$ if $N(f, g) > 0$ [5].

Let M, N be two connected compact manifolds. We say that the pair (M, N) has *the Jiang type property for coincidences* if for any maps $f, g : M \rightarrow N$, $N(f, g) = 0 \Leftrightarrow R(f, g) = \infty$ and $N(f, g) = R(f, g)$ if $N(f, g) > 0$.

By Theorem 2.3, we see that any pair (M, N) of compact nilmanifolds has the Jiang type property for coincidences (even if $\dim M < \dim N$).

The proof of Corollary 2.6 in fact extends to the following general fact.

Proposition 3.1. Suppose M has a regular finite cover \hat{M} such that (\hat{M}, N) has the Jiang type property for coincidences. Then the pair (M, N) also has the Jiang type property for coincidences.

In the sequel, we investigate the Jiang type property for coincidences. In particular, we explore conditions so that the pair (M, N) possesses such property where N is a compact infra-nilmanifold.

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DEPARTAMENTO DE MATEMÁTICA - IME - UNIVERSIDADE DE SÃO PAULO,, RUA DO MATÃO, 1010, CEP 05508-090
- SÃO PAULO - SP - BRAZIL

E-mail address: `dlgoncal@ime.usp.br`

DEPARTMENT OF MATHEMATICS, BATES COLLEGE, LEWISTON, ME 04240, U.S.A.

E-mail address: `pwong@bates.edu`